

Universality in weak chaos

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We consider a general class of intermittent maps designed to be weakly chaotic, i.e., for which the separation of trajectories of nearby initial conditions is weaker than exponential. We show that all its spatio-temporal properties, hitherto regarded independently in the literature, can be represented by a single characteristic function ϕ . A universal criterion for the choice of ϕ to be fulfilled by weakly chaotic systems is obtained via the Feigenbaum's renormalization-group approach. We find a general expression for the dispersion rate of initially nearby trajectories and we show that the instability scenario for weakly chaotic systems is more general than that originally proposed by Gaspard and Wang [Proc. Natl. Acad. Sci. USA **85**, 4591 (1988)].

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The instability of deterministic motion is typically characterized by the growing separation of initially nearby trajectories, which, in turn, indicates that dynamics is high sensitive to its initial state; very small differences at one moment in such systems can result in very large differences later on. For a chaotic dynamical system $x_t = T^t(x_0)$, the separation of initially nearby trajectories, namely $\delta x_t = T^t(x_0 + \delta x_0) - T^t(x_0)$, evolves asymptotically as

$$|\delta x_t| \sim |\delta x_0| \exp[\Lambda_t(x_0)\zeta(t)], \quad (1)$$

for almost every choice of δx_0 within a certain neighborhood of x_0 . The positive coefficient $\Lambda_\infty(x_0)$ stands for the largest Lyapunov exponent since $\zeta(t) \sim t$, and most of well-known chaotic systems are ruled by dispersion rates $\zeta(t)$ of this type [1–3]. After the pioneering work of Gaspard and Wang [4], there has been a growing interest in recent years in understanding the so-called weakly chaotic (or sporadic) systems, for which the separation of initially nearby trajectories is weaker than exponential, i.e. $\zeta(t)/t \rightarrow 0$ [5–10]. Such systems also exhibit unpredictable behavior although they have zero Lyapunov exponent in the usual sense.

In this paper we consider a general class of maps of the interval, which will hopefully be weakly chaotic, and we tie all of its spatio-temporal properties, including the map equations itself, to a single characteristic function ϕ . The first step towards a unifying framework is based on determining the eigenfunctions of the Feigenbaum's doubling operator [11]

$$\mathcal{F}g(x) = \alpha g(g(x/\alpha)), \quad (2)$$

where α is a rescaling factor. The eigenfunction that outlines the universality class is a fixed point of \mathcal{F} , i.e. $\mathcal{F}g_* = g_*$. We shall see that

$$g_*(x) = \phi^{-1}[\phi(x) - \tau], \quad (3)$$

where ϕ^{-1} denotes the inverse of ϕ and τ is a constant.

As far as we know, the weakly chaotic behavior $\zeta(t)/t \rightarrow 0$ results from the intermittent switching between long regular phases (so-called laminar) and short irregular bursts [4–10]. Since the early 1980's, the idea that the same functional equation employed by Feigenbaum for studying the period-doubling cascade can also be used to describe intermittency and dissipative systems has been developed by many authors, notably [12–16]. From a viewpoint more connected to the intermittency phenomenon, considerations done so far about universality have been based on the scaling properties of the laminar length [12–15]. Our results also shed some light on the scaling hypothesis behind the relationship between laminar length and Feigenbaum's doubling operator (2).

We provide a general formula for the dispersion rate $\zeta(t)$, one of the main results of this manuscript. But besides an attempt to outline the subexponential instability in weak chaos, our goal is to provide a full description of a nonlinear dynamical system whenever a scaling property is present. Here we have two key quantities determining the spatio-temporal properties of weakly chaotic systems. The invariant density $\rho(x)$ gives us the measure of concentration trajectories at each stage of intermittency, whereas the residence times at each ones are ruled by the waiting-time probability density function $\psi(t)$ of the laminar phase. By introducing a proper modelling of intermittency mechanism, together with a renormalization-group approach, we establish a universal criterion for the choice of ϕ which enables us to predict the dispersion rate $\zeta(t)$, as well as determining $\rho(x)$ and $\psi(t)$. We will also see that these results enable us to predict the anomalous subdiffusion for spatially extended versions of such systems by means of a relationship between $\zeta(t)$ and the mean squared displacement $\sigma^2(t)$.

Let us consider the general class of piecewise expanding maps, from $[0, 1]$ to itself, in the form $x_{t+1} = T(x_t)$ so that

$$T(x) = x + f(x), \quad (4)$$

with a marginal fixed point at $x = 0$, i.e., $f(x \rightarrow 0) = 0$ and $f'(x \rightarrow 0) = 0$. Although the global form of T is less

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relevant here, the mechanism of intermittency also relies on the existence of a interval point c such that $T(c_-) = 1$ and $T(c_+) = 0$, ensuring the chaotic reinjection of trajectories from $(c, 1]$ to the laminar region near the marginal point. One key ingredient for the resulting weakly chaotic behavior is how invariant density of T behaves near the marginal fixed point. First of all, $\rho(x)$ must be an eigenfunction of the Perron-Frobenius operator [17]

$$\mathcal{P}\rho(x) = \sum_j \frac{\rho[T_j^{-1}(x)]}{|T'[T_j^{-1}(x)]|}, \quad (5)$$

where the sum extends over all preimages T_j^{-1} of the point x at which the density is to be computed. Near the marginal fixed point $x = 0$ we have $T_0^{-1}(x) \sim x$, whereas $T_{j \neq 0}^{-1}(x)$ assumes values on the interval far away from zero. Then one has $\rho(x) \sim \rho(x)[1 - f'(x)] + \text{constant}$, resulting in the divergent invariant density

$$\rho(x) \sim \frac{1}{f'(x)} \quad (6)$$

near $x = 0$, up to a positive multiplicative constant for non-normalizable $\rho(x)$.

The finite-time (generalized) Lyapunov exponent of the map T satisfying Eq. (1) is

$$\Lambda_t(x_0) = \frac{1}{\zeta(t)} \sum_{k=0}^{t-1} \ln |T'[T^k(x_0)]|. \quad (7)$$

In order to obtain the dispersion rate $\zeta(t)$, we expect that the average of Lyapunov exponent (7) over the initial condition ensemble may be estimated by using a density function $\rho(x, t)$ so that

$$\langle \Lambda_\infty \rangle = \frac{1}{\zeta(t)} \int_0^t du \int_0^1 dx \rho(x, u) \ln |T'(x)|, \quad (8)$$

as $t \rightarrow \infty$. To find $\rho(x, t)$ we will consider the continuous-time stochastic model proposed in [18],

$$\frac{\partial}{\partial t} \rho(x, t) = -\frac{\partial}{\partial x} [v(x) \rho(x, t)] + C(t). \quad (9)$$

The convective equation (9) describes the stochastic motion of a particle initially in the laminar phase until it is random reinjected back to a position on the interval after reaching (crossing) the point $x = c$. The term $v(x)$ describes the particle's velocity in the laminar phase, and the reinjection source term $C(t)$ is chosen to fulfill conservation of quantity $\int_0^c dx \rho(x, t)$. Equation (9) can be completely solved using the method of characteristics [19]. Assuming uniform initial density $\rho(x, 0) = 1$ and considering the auxiliary functions $\phi(x)$ and $\Psi(x, t)$ as follow

$$-\int \frac{1}{v(x)} dx = \phi(x), \quad \Psi(x, t) = -\frac{\partial}{\partial t} \phi^{-1}[\phi(x) + t], \quad (10)$$

we get the general solution for the Laplace transform $\mathcal{L}_s[\rho(x, t)] = \tilde{\rho}(x, s)$:

$$\tilde{\rho}(x, s) = \frac{1}{v(x)} \frac{\tilde{\Psi}(x, s)}{1 - \tilde{\Psi}(c, s)}. \quad (11)$$

The corresponding waiting-time density function $\psi(t)$ is given by $\psi(t) \sim \Psi(c, t)$. Note that ϕ in Eq. (10) is defined up to an additive constant, to be suitably chosen so that $\tilde{\Psi}(c, s \rightarrow 0) = 1$ due to normalization of $\psi(t)$. Thus we set $\phi(c) = 1$, and therefore we have (hereafter omitting positive multiplicative constants)

$$\psi(t) \sim -[\phi^{-1}(t+1)]'. \quad (12)$$

The convective model (9) fulfills the ergodic properties of our map T noting that the average (8) requires that $\rho(x, t) \sim \zeta'(t)\rho(x)$. Upon considering this asymptotic behavior, we must have $v(x) \sim x/\rho(x)$. The dispersion rate $\zeta(t)$ can now be calculated from the general solution (11), and one gets

$$\zeta(t) \sim \mathcal{L}_t^{-1} \left\{ \frac{1}{s[1 - \tilde{\psi}(s)]} \right\}. \quad (13)$$

Later we will solve Eq. (13) explicitly in terms of ϕ . Besides the waiting-time density (12), the map T and its corresponding invariant density can also be put forward here in terms of ϕ as follow

$$T'(x) \sim 1 - 1/x\phi'(x), \quad \rho(x) \sim -x\phi'(x). \quad (14)$$

Now we are ready to carry out the renormalization-group approach by using the length of laminar motion x_* , which is usually known for its scaling properties [12–15, 20]. A detailed justification for this choice will be given here later on. From the definition of ϕ in Eq. (10) we have the relation $\tau = \phi(x) - \phi(x_*)$, τ being the time interval between laminar positions x_* and x . Thus the eigenfunction of doubling operator (2) is $g_*(x) = x_*$, as presented before in Eq (3). The boundary conditions must be such that $\phi'(x \rightarrow 0) = -\infty$, since we must have $T'(x \rightarrow 0) = 1$ and positive invariant density. Then one has $g_*(x) \sim x$, so that $g_*(x)$ has the same characteristics of map T for x near zero, i.e.

$$g_*(x \rightarrow 0) = 0, \quad g'_*(x \rightarrow 0) = 1. \quad (15)$$

For our first example, first notice that the eigenfunction $g_*(x)$ of operator (2) can be recast in the form $g_*(x) = \alpha\phi^{-1}[\phi(x/\alpha) - 2\tau]$. Introducing the auxiliary functions $h_1(x) = \alpha\phi^{-1}(2x)$ and $h_2(x) = \phi(x/\alpha)/2$, one has $g_*(x) = h_1[h_2(x) - \tau]$, admitting as possible solutions $h_1 = \phi^{-1}$ and $h_2 = \phi$. Therefore, the characteristic function ϕ may satisfy the relations $\phi(x/\alpha) = 2\phi(x)$ and $\alpha\phi^{-1}(2x) = \phi^{-1}(x)$. The first of latter two equations was derived in a different way in [14], and it is easy to see that the pair admits as solution

$$\phi(x) = x^{-1/\gamma}, \quad \alpha = 2^\gamma, \quad (16)$$

for $\gamma > 0$. This result gives us the well-known class of Pomeau-Manneville maps, for which $f(x) \sim x^{1+1/\gamma}$ near $x = 0$. From the physical point of view, the original Pomeau-Manneville system, i.e. $\gamma = 1$, is paradigmatic since it corresponds to certain Poincaré sections related to the Lorenz attractor [21]. Lastly, one has $\rho(x) \sim x^{-1/\gamma}$ and $\psi(t) \sim (t+1)^{-(1+\gamma)}$. Our Eq. (13) gives us the corresponding weakly chaotic regimes:

$$\zeta(t) \sim \begin{cases} t^\gamma, & 0 < \gamma < 1, \\ t(\ln t)^{-1}, & \gamma = 1, \end{cases} \quad (17)$$

and also $\zeta(t) \sim t$ for $\gamma > 1$.

The weakly chaotic instability $\zeta(t)/t \rightarrow 0$ stems from the diverging behavior of invariant measure μ near the marginal fixed point, e.g. $0 < \gamma \leq 1$ for the class of Pomeau-Manneville maps (16). From Eq. (14) we have the invariant measure $\mu([x, y]) \sim x\phi|_y^x - \int_y^x du\phi(u)$. Therefore, the divergence of invariant measure occurs provided that the characteristic function ϕ obeys

$$\int dx\phi(x) \rightarrow -\infty \quad (18)$$

as $x \rightarrow 0$, which is more restrictive than the boundary conditions (15). From Eq. (12) we also have the relation $\int dt\psi(t)t \sim -\int dx\phi(x)$, and therefore Eq. (18) implies the divergence of mean waiting time. Thus, the weakly chaotic behavior occurs provided that $\psi(t)$ does not decrease faster than t^{-2} . In such cases, Eq. (13) can be solved by making use of Karamata's Abelian and Tauberian theorems for the Laplace-Stieltjes transform [22]. By considering the general form of cumulative distribution function associated to $\psi(t)$, i.e., $\int_0^t du\psi(u) \sim 1 - 1/l(t)t^\gamma$ and $l(t)$ being a slowly varying function at infinity, one obtains from Eq. (13)

$$\zeta(t) \sim \int_0^t du \mathcal{L}_u^{-1} \left[s \frac{l(1/s)}{s^{1+\gamma}} \right] \sim l(t)t^\gamma \quad (19)$$

noting that for $\gamma = 1$ we must have $l(t \rightarrow \infty) = 0$, thus excluding only the $\gamma = 1$ border case $\psi(t) \sim t^{-2}$ (corresponding to $\phi(x) \sim x^{-1}$). Furthermore, noting that $\int_0^t du\psi(u) \sim 1 - \phi^{-1}(t)$, one finally gets

$$\zeta(t) \sim \frac{1}{\phi^{-1}(t)}. \quad (20)$$

The result (20) enables us to develop models with weakly chaotic behavior provided that the criterion (18) is fulfilled. Thus, our second example relies on the family of maps for which $\phi(x)$ behaves as

$$\phi(x) \sim \phi_0(x) \exp(x^{-\beta}) \quad (21)$$

for $\beta > 0$, provided $\phi_0(x)$ does not go to zero as fast as $\exp(-x^{-\beta})$ for $x \rightarrow 0$. This model should be understood as a $\gamma \rightarrow 0$ limiting case for the rescaling factor $\alpha = 2^\gamma$. Using again the machinery we have developed, one has

$\psi(t) \sim t^{-1}(\ln t)^{-(1+1/\beta)}$, irrespective of ϕ_0 , resulting in the strong anomaly dispersion rate

$$\zeta(t) \sim \ln^{1/\beta} t, \quad (22)$$

which also agrees with Eq. (20). The dispersion rates (17) and (22) are in perfect agreement with their corresponding quantities in the infinite ergodic theory [23, 24], the so-called return sequences a_t [25]. Such sequences ensure a suitable time-weighted average of observables that converge in distribution terms towards a Mittag-Leffler distribution [25]. Note also that our Eq. (6) generalizes the invariant densities obtained in [26] for these types of systems. Lastly we observe that, under condition (18), the dispersion rate (20) shows that the instability scenario for weakly chaotic systems is more general than that originally proposed by Gaspard and Wang in [4].

Yet another application of the renormalization-group approach: weakly chaotic maps of the type (4) have been extensively used in the literature to model systems that exhibit anomalous transport, see for instance [10] and references therein. The mechanism for generating deterministic subdiffusion is based on the extended version of map (4), from $[0, 1/2]$ to the entire real line, according to the rules $f(x+N) = f(x) + N$ and $f(-x) = -f(x)$, where N assume integer values. This results in a series of lattice cells with marginal points located at $x = N$. The corresponding transport properties can be understood in terms of a continuous-time random walk picture of this model, with probability density of waiting times $\psi(t)$ near each marginal point. The mean squared displacement $\sigma^2(t)$ for such model is given by [27]

$$\mathcal{L}_s\{\sigma^2(t)\} \sim \frac{\tilde{\psi}(s)}{s[1 - \tilde{\psi}(s)]}. \quad (23)$$

Since $\tilde{\psi}(s \rightarrow 0) = 1$, from Eq. (13) one has

$$\sigma^2(t) \sim \zeta(t). \quad (24)$$

The mean squared displacements for the extended versions of models (16) and (21) based on our results are in perfect agreement with those obtained respectively in [28] and [29, 30]. Equation (24) is particularly interesting because it is a non-trivial extension for weakly chaotic systems of a relationship typically observed in usual chaos, namely $\sigma^2(t) \sim \zeta(t) \sim t$.

Why the laminar length x_* does scale according to the Feigenbaum's doubling operator? The extension of the doubling operator for the class of systems discussed here can be understood by means of the scaling limit

$$g(x) = \lim_{n \rightarrow \infty} \alpha^n T^{2^n}(x/\alpha^n), \quad (25)$$

since we have $T^n(x) \sim x + nf(x)$ near zero. By using recursion, it is simple to see that Eq. (25) leads to the doubling operator (2) with $g_*(x) \sim x + \text{const.}f(x)$ and $\alpha = 2^\gamma$. It is important to emphasize here that we need not find eigenfunctions of Eq. (2) covering the whole

interval $[0, c]$, but just a vicinity of zero. Thus, we can expand our proposal eigenfunction (3) for $\tau \approx 0$ since $\phi(x)$ is singular at $x = 0$, resulting $g_*(x) \sim x - \tau/\phi'(x)$. Now, from Eq. (14), one has $xf' \sim -1/\phi'$, and our scaling hypothesis boils down simply to $xf' \sim f$.

We close by emphasizing that the intermittent behavior is an ubiquitous phenomenon in nonlinear science. Different types of intermittency (with or without external noise) have been studied and classified as types I-III intermittences [21, 31], crisis-induced intermittency [32], on-off intermittency [33, 34], eyelet intermittency [35–37], and ring intermittency [38]. Despite their different characteristics, they all have in common the presence of some kind of scaling behavior. Under certain circumstances, some types of intermittency may even be equivalent, such

as the type-I intermittency in the presence of noise and eyelet intermittency observed in the onset of phase synchronization [39]. We believe that a major challenge for further works would be a general characterization of intermittency phenomenon having the renormalization-group approach as a unifying framework.

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